# Observing through the Turbulent Atmosphere

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# Plan of this Talk

- A few useful results from Fourier theory
- Motivation of the Kolmogorov model for turbulence
- Statistical description of Kolmogorov turbulence
- Wave propagation through turbulence
- Image formation by telescopes and the effect of turbulence on images
- Useful parameters that describe turbulence

# Convolution, Correlation, and Structure Function

- Convolution: $g * h \equiv \int_{-\infty}^{\infty} g(t-\tau)h(\tau)d\tau$
- Correlation: $Corr(g,h) \equiv \int_{-\infty}^{\infty} g(t+\tau)h(\tau)d\tau$
- Covariance: $B_g \equiv Corr(g,g)$
- Structure function: $D_g(t_1,t_2) \equiv \left\langle \left|g(t_1)-g(t_2)\right|^2 \right\rangle$
- If $g$ describes a homogeneous and isotropic process, $D_g$ depends only on $t \equiv |t_1 - t_2|$, and

$$D_g(t) = 2\left[B_g(0) - B_g(t)\right]$$

# A few Important Results from Fourier Theory

- Convolution theorem

$$g * h \;\Leftrightarrow\; G(f)\cdot H(f)$$

- Correlation theorem

$$Corr(g,h) \;\Leftrightarrow\; G(f)\cdot H^{*}(f)$$

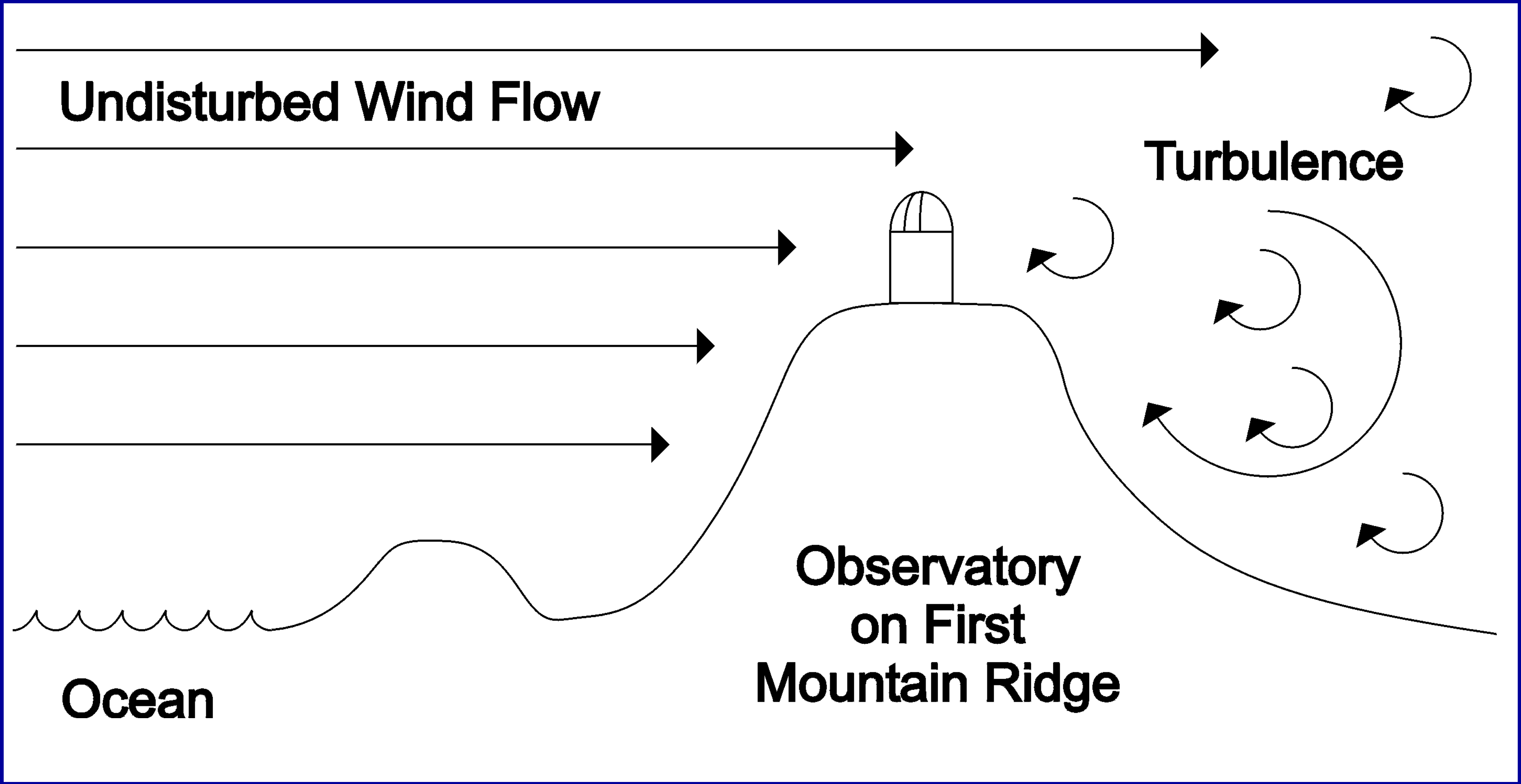
- Wiener-Khinchin theorem

$$Corr(g,g) \;\Leftrightarrow\; \left|G(f)\right|^{2}$$

- Parseval's theorem

$$\text{Total Power} \equiv \int_{-\infty}^{\infty} \left|h(t)\right|^{2} dt = \int_{-\infty}^{\infty} \left|H(f)\right|^{2} df$$

# Turbulence Generation



# The Kolmogorov Turbulence Model

- For atmospheric flows the Reynolds number $Re \equiv VL / v \geq 10^6 \;\; \Rightarrow$ flow is highly turbulent
- Turbulent energy is generated on large scale $L_0$, dissipated on small scale $l_0$
- $L_0$ is called *outer scale*, $l_0$ is called *inner scale*
- In the *inertial range* between $l_0$ and $L_0$, there is a universal description for the turbulence spectrum
- The only two relevant parameters are the rate of energy generation $\varepsilon$ and the kinematic viscosity $v$

# The Structure Function of Kolmogorov Turbulence

- The units of $\nu$ are $m^2s^{-1}$, the dimensions of $\varepsilon$ are $J\,s^{-1}kg^{-1} = m^2s^{-3}$
- The velocity structure function can be written as

$$D_v(R_1, R_2) \equiv \left\langle \left| v(R_1) - v(R_2) \right|^2 \right\rangle$$
$$= \alpha \cdot f\left(\left|R_1 - R_2\right| / \beta\right)$$

- The dimensions of $\alpha$ are $m^2s^{-2}$ $\Rightarrow \alpha = \nu^{1/2}\varepsilon^{1/2}$
- The dimensions of $\beta$ are m $\Rightarrow \beta = \nu^{3/4}\varepsilon^{-1/4}$

# Completion of Dimensional Analysis

- In the inertial range dissipation plays no role
  $\Rightarrow$ dependence on $v$ must drop out
- This is possible only if $f = k \cdot \left( |R_1 - R_2| / \beta \right)^{2/3}$
- We can therefore write
  $$D_v(R_1, R_2) = C_v^2 |R_1 - R_2|^{2/3}$$
- The constant $C_v^2$ describes the strength of the turbulence

# Structure Function of the Refractive Index

- Turbulence carries "parcels" of air with different temperature
- The parcels are in pressure equilibrium and thus have different density and index of refraction
- The corresponding structure functions are

$$D_T(R_1, R_2) = C_T^2 |R_1 - R_2|^{2/3} \quad \text{and}$$

$$D_N(R_1, R_2) = C_N^2 |R_1 - R_2|^{2/3} \quad \text{with}$$

$$C_N = \left(78 \cdot 10^{-6}\, p[\text{mbar}] / T^2[\text{K}]\right) \cdot C_T$$

# The Power Spectrum of the Refractive Index

- The structure function $D$ is related to the covariance $B$ by $D(R) = 2[B(0) - B(R)]$
- The covariance $B$ is the Fourier transform of the power spectral density $\Phi$ (Wiener-Khinchin theorem)
- We can thus compute $\Phi$ from $D$
- For Kolmogorov turbulence the result is

$$\Phi(\kappa) = 0.0365\, C_N^2\, \kappa^{-5/3}$$

# The Effect of a Turbulent Layer

- We look at the propagation of a wavefront $\psi(x) = \exp[i\phi(x)]$ through a turbulent layer of thickness $\delta h$ at height $h$
- The phase shift produced by refractive index fluctuations is

$$\phi(x) = k \int_h^{h+\delta h} dz \; n(x,z)$$

# The Coherence Function of the Wavefront

- We will need the *coherence function* of the wavefront

$$
\begin{aligned}
B_h(r) &\equiv \left\langle \psi(x+r)\psi^*(x) \right\rangle \\
&= \left\langle \exp i\left[\phi(x) - \phi(x+r)\right]\right\rangle \\
&= \exp\left[-\tfrac{1}{2}\left\langle \left|\phi(x) - \phi(x+r)\right|^2 \right\rangle\right] \\
&= \exp\left[-\tfrac{1}{2} D_\phi(r)\right]
\end{aligned}
$$

- Next goal: calculate $D_\phi(r)$

# Expectation Value of Exponential

- Let $\chi$ be a Gaussian variable with zero mean and variance $\sigma^2$
- $$\begin{aligned}\langle \exp(\alpha\chi) \rangle &\equiv \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\chi \exp(\alpha\chi) \exp\left( -\frac{\chi^2}{2\sigma^2} \right) \\ &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\chi \exp\left[ -\tfrac{1}{2} \left( \frac{\chi^2}{\sigma^2} - 2\alpha\chi \right) \right] \\ &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\chi \exp\left[ -\frac{1}{2\sigma^2} \left( \chi - \alpha\sigma^2 \right)^2 \right] \cdot \exp\left( \tfrac{1}{2}\alpha^2\sigma^2 \right) \\ &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\tilde{\chi} \exp\left( -\frac{\tilde{\chi}^2}{2\sigma^2} \right) \cdot \exp\left( \tfrac{1}{2}\alpha^2\sigma^2 \right) \\ &= \exp\left( \tfrac{1}{2}\alpha^2 \left\langle \chi^2 \right\rangle \right)\end{aligned}$$

# Phase Covariance

- With $z \equiv z' - z''$

$$B_\phi(r) \equiv \langle \phi(x)\phi(x+r) \rangle$$

$$= k^2 \int_h^{h+\delta h} \int_h^{h+\delta h} dz' dz'' \langle n(x,z') n(x+r,z'') \rangle$$

$$= k^2 \int_h^{h+\delta h} dz' \int_{h-z'}^{h+\delta h - z'} dz \, B_N(r,z)$$

- For $\delta h$ much larger than the correlation scale

$$B_\phi(r) = k^2 \delta h \int_{-\infty}^{\infty} dz \, B_N(r,z)$$

# Computation of the Phase Structure Function

- $$D_\phi(r) = 2\left[B_\phi(0) - B_\phi(r)\right]$$
  $$= 2k^2 \delta h \int_{-\infty}^{\infty} dz \left[B_N(0,z) - B_N(r,z)\right]$$
  $$= 2k^2 \delta h \int_{-\infty}^{\infty} dz$$
  $$\left[\left(B_N(0,0) - B_N(r,z)\right) - \left(B_N(0,0) - B_N(0,z)\right)\right]$$
  $$= k^2 \delta h \int_{-\infty}^{\infty} dz \left[D_N(r,z) - D_N(0,z)\right]$$
  $$= k^2 \delta h C_N^2 \int_{-\infty}^{\infty} dz \left[\left(r^2 + z^2\right)^{1/3} - z^{2/3}\right]$$
  $$= 2.914\, k^2 \delta h\, C_N^2\, r^{5/3}$$

# The Wavefront Coherence Function

- For the layer under consideration we have obtained

$$B_h(r) = \exp\left[-\tfrac{1}{2}\left(2.914\, k^2 C_N^2\, \delta h\, r^{5/3}\right)\right]$$

- Integration over the whole atmosphere, and taking the zenith angle $z$ into account gives

$$B(r) = \exp\left[-\tfrac{1}{2}\left(2.914\, k^2 (\sec z)\, r^{5/3} \int dh\, C_N^2(h)\right)\right]$$

# The Fried Parameter $r_0$

- We define the *Fried parameter* $r_0$ by

$$r_0 \equiv \left[0.423 k^2 (\sec z) \int dh \, C_N^2(h)\right]^{-3/5}$$

- Now we can write

$$B(r) = \exp\left[-3.44 \left(\frac{r}{r_0}\right)^{5/3}\right] \qquad \text{and}$$

$$D_\phi(r) = 6.88 \left(\frac{r}{r_0}\right)^{5/3}$$

# Fraunhofer Diffraction

- The complex amplitude $A$ of a wave $\psi$ diffracted at an aperture $P$ with area $\Pi$ can be computed from Huygens' principle
- In the far field $A$ is thus given by
  $$A(\alpha) = \frac{1}{\sqrt{\Pi}} \int dx\, \psi(x) P(x) \exp[-2\pi i \alpha x / \lambda]$$
- With $u \equiv x / \lambda$ we can write
  $$A(\alpha) = \frac{1}{\sqrt{\Pi}} FT[\psi(u) P(u)]$$

# Optical Image Formation

- The intensity distribution in the focal plane (point spread function) is

$$S(\alpha) = \left|A(\alpha)\right|^2 = \tfrac{1}{\Pi}\left|FT\left[\psi(u)P(u)\right]\right|^2$$

- From the Wiener-Khinchin theorem we get

$$\left\langle S(f)\right\rangle = \tfrac{1}{\Pi}\int du \left\langle \psi(u)\psi^*(u+f)\right\rangle P(u)P^*(u+f)$$

$$= B(f)T(f)$$

- Here we have introduced the *telescope transfer function*

$$T(f) = \tfrac{1}{\Pi}\int du\, P(u)P^*(u+f)$$

# Resolving Power and Diffraction-Limited Images

- We define the *resolving power R* by

$$R \equiv \int df\, B(f)\, T(f)$$

- In the absence of turbulence, $B = 1$, and

$$R = R_{\text{tel}} = \int df\, T(f)$$
$$= \frac{1}{\Pi} \iint du\,df\, P(u)\, P^*(u+f)$$
$$= \frac{1}{\Pi} \left| \int du\, P(u) \right|^2 = \frac{\pi}{4} \left( \frac{D}{\lambda} \right)^2$$

- The last equality holds for a circular aperture

# Seeing-Limited Images

- For strong turbulence $T = 1$ in the region where $B$ is non-zero, and

$$R = R_{\mathrm{atm}} = \int df\, B(f)$$

$$= \int df\, \exp\left[-Kf^{5/3}\right]$$

$$= \frac{6\pi}{5}\Gamma\left(\frac{6}{5}\right)K^{-6/5} = \frac{6\pi}{5}\Gamma\left(\frac{6}{5}\right)\cdot\left[3.44\left(\frac{\lambda}{r_0}\right)^{5/3}\right]^{-6/5}$$

$$= \frac{\pi}{4}\left(\frac{r_0}{\lambda}\right)^2$$

# Significance of the Fried Parameter $r_0$

- The resolution of long exposures through the atmosphere is the same as the resolution obtained with a telescope of diameter $r_0$
- The phase variance over an aperture with diameter $r_0$ is approximately 1 rad$^2$
- $r_0$ depends on the turbulence profile $C_N{}^2(h)$, the zenith angle $z$, and the observing wavelength $\lambda$
- The wavelength dependence is $r_0 \propto \lambda^{6/5}$; this leads to an image size ("seeing") $\alpha \propto \lambda / r_0 \propto \lambda^{-1/5}$

# Typical Value of $r_0$

- At good sites and under good conditions, $r_0$ at 500 nm is typically in the range 10…20 cm
- This corresponds to an image size of 0.5″ … 1″
- At any site, the night-to-night variations of $r_0$ are large
- There are also short-term fluctuations on all time scales, which complicate the calibration of high-resolution observations

# The Strehl Ratio

- The quality of an imaging system is often measured by the *Strehl ratio S*, defined as the on-axis intensity in the actual image divided by the peak intensity in a diffraction-limited image
- For phase errors ≤ 2 rad, $S \approx \exp\left[-\sigma_\phi^2\right]$
- The Hubble Space Telescope optics have $S \approx 0.1$ (without corrective optics)
- A telescope with diameter $r_0$ delivers $S = 0.37$ (without tip-tilt correction)

# Practical Consequences of Non-Perfect Strehl Ratio

- If $S \geq 0.1$, image deconvolution software can usually be used to obtain diffraction-limited images, but the dynamic range is limited
- In an interferometer the visibility cannot be better than $S$
- The coupling efficiency into single-mode fibers is approximately equal to $S$
- For telescopes with size up to $\sim 3r_0$ tip-tilt correction can dramatically improve $S$

# Taylor's "Frozen Turbulence" Hypothesis and $\tau_0$

- It is frequently assumed that the time constant for changes in the turbulence pattern is much longer than the time it takes the wind to blow the turbulence past the telescope aperture
- Atmospheric turbulence is often dominated by a single layer
- The temporal behavior of the turbulence can therefore be characterized by a time constant $\tau_0 \equiv r_0 / v$, where $v$ is the wind velocity in the dominant layer

# Short and Long Exposures

- Observations with exposure time $t \ll \tau_0$ (so-called "short exposures") produce images through one instantaneous realization of the atmosphere ("speckle images")
- Long exposures with $t \gg \tau_0$ average over the atmospheric random process
- In an interferometer $\tau_0$ sets the time scale for detector readout or fringe tracking

# Phase Variance between Rays from two Stars

- The rays "from" the telescope "to" two stars separated by an angle $\theta$ coincide at the pupil; at a distance d their separation is $r = \theta d = \theta h \sec z$
- We insert this relation in

$$\left\langle \left|\phi(0) - \phi(r)\right|^2 \right\rangle = D_\phi(r) = 2.914 k^2 \sec z \delta h C_N^2 r^{5/3}$$

- The result is

$$\left\langle \sigma_\phi^2 \right\rangle = 2.914 k^2 \sec z \int dh\, C_N^2(h) \left(\theta h \sec z\right)^{5/3}$$

$$= 2.914 k^2 (\sec z)^{8/3} \theta^{5/3} \int dh\, C_N^2(h) h^{5/3}$$

# Angular Anisoplanatism

- We define

$$\theta_0 \equiv \left[ 2.914 k^2 (\sec z)^{8/3} \int dh\, C_N^2(h) h^{5/3} \right]^{-3/5}$$

- Now we can write

$$\left\langle \sigma_\phi^2 \right\rangle_{\text{aniso}} = \left( \frac{\theta}{\theta_0} \right)^{5/3}$$

- Anisoplanatism is dominated by high layers
- The short-exposure point spread functions for two stars separated by more than $\theta_0$ are different, but the long-exposure psf's are (nearly) identical

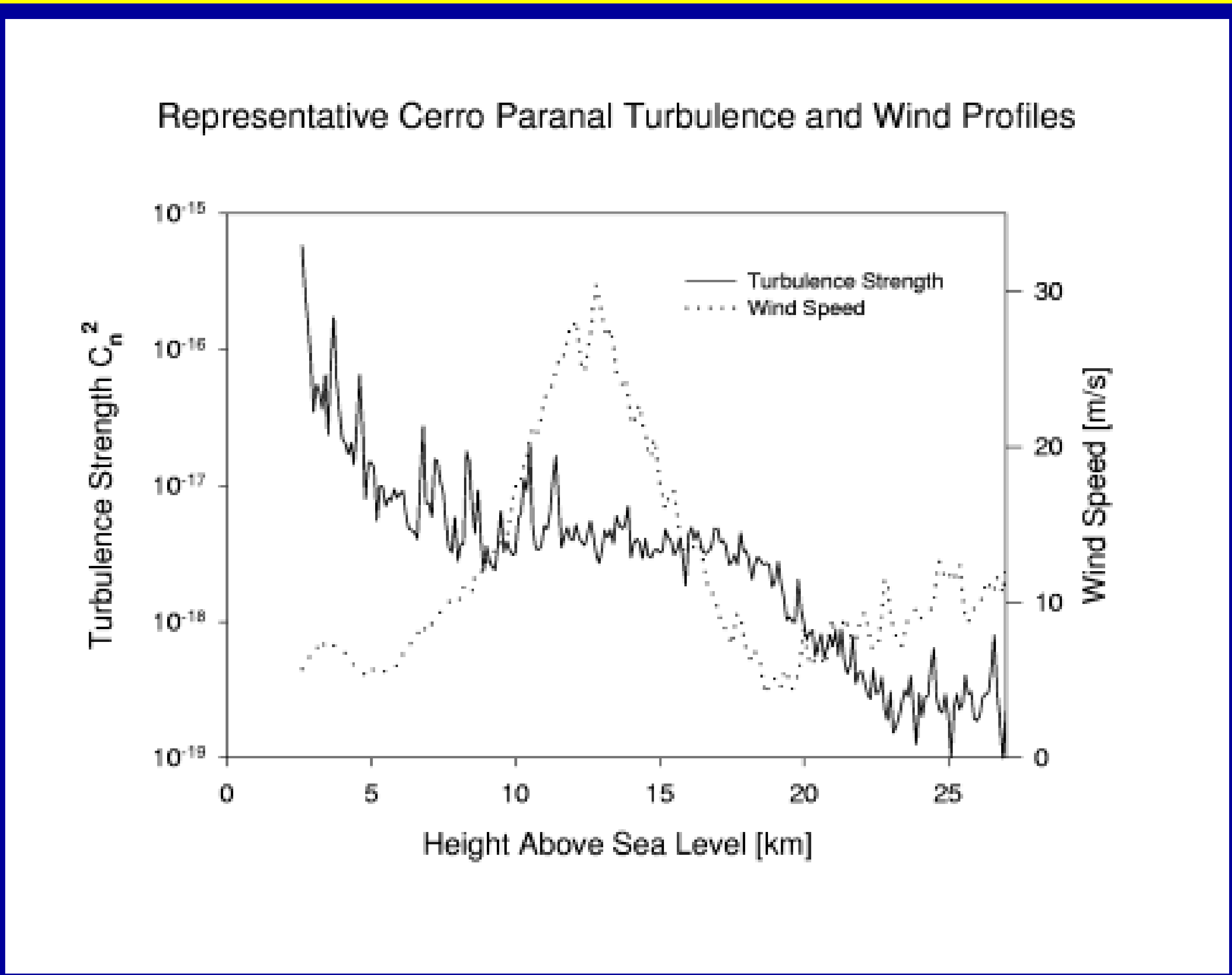
# Fresnel Length and Diffraction Effects

- The geometric optics approximation of propagation is only valid for paths shorter than the *Fresnel propagation length* $d_F \equiv r_0^2 / \lambda$
- For $r_0 = 10$ cm, $\lambda = 500$ nm, the Fresnel length is 20 km
- The geometric approximation is therefore a good first-order approach, but diffraction is not negligible, especially at short wavelengths, large zenith angles, and poor observing sites

# Scintillation

- Diffraction gives rise to *scintillation*, i.e., intensity fluctuations that are important for photometry if the exposure time is short
- The local intensity fluctuations are given by
  $$\sigma^2_{\ln I} = 2.24 k^{7/6} (\sec z)^{11/6} \int dh\, C_N^2(h) h^{5/6}$$
- Scintillation is dominated by high-altitude turbulence
- For telescopes larger than the Fresnel scale $r_F \equiv \sqrt{\lambda h \sec z}$, aperture averaging is important

# Turbulence Profiles



# Consequences for Optical / Infrared Interferometry

- Optical interferometers with telescopes larger than $r_0$ need adaptive optics (tip-tilt correction only is sufficient up to $3r_0$)
- The search radius for dual-star interferometry is limited by anisoplanatism
- Null depth is limited by atmospheric residuals
- Fringe tracking servos or detector readouts must work on the atmospheric time scale

# Atmospheric Phase Noise and Fringe Tracking

- The power spectrum of atmospheric phase fluctuations (Kolmogorov approximation) is
$$\Phi_\phi(f) = 0.077\,\tau_0^{-5/3} f^{-8/3}$$
- If $H(f)$ is the closed-loop fringe servo transfer function, the residual phase variance is given by
$$\sigma_R^2 = \int_0^\infty df \left|1 - H(f)\right|^2 \Phi_\phi(f)$$
- Define the *Greenwood frequency* by
$$f_G \equiv \left[0.102\,k^2 (\sec z) \int dh\, C_N^2(h)\, v^{5/3}(h)\right]^{3/5}$$

# Null Depth in the Presence of Phase Noise

- In many cases, the null depth due to residual phase fluctuations can be written as

$$\overline{N} = \frac{1}{4}\sigma_R^2 = \frac{1}{2}\kappa\left(\frac{f_G}{f_S}\right)^{5/3}$$

- $\kappa = 0.191$ for sharp cutoff, $\kappa = 1$ for RC filter, $\kappa = 28.4$ for pure delay (loop lag)
- Detailed modeling of fringe tracking servo needed

# Numerical Estimate of Fringe Tracking Residuals

- Assume a servo with a 2 ms “pure delay”
- Assume $f_G = 21.35$ Hz at $\lambda = 500$ nm (corresponds to $r_0 = 20$ cm, $v = 10$ m/s)
- For these parameters the null depth is $6.3 \cdot 10^{-3}$ at K band
- The null depth due to high-frequency fringe tracking residuals scales with $\lambda^{-5/3}$

# Literature References

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